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The traditional nuclear pairing problem is shown to be in one-to-one correspondence with a classical electrostatic problem in two dimensions. We make use of this analogy in a series of calculations in the Tin region, showing that the extremely rich phenomenology that appears in this classical problem can provide interesting new insights into nuclear superconductivity.

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Pairing is a pervasive feature in nuclear structure. Perhaps its most dramatic manifestation is nuclear superconductivity, the analogue of the more familiar superconductivity that arises due to the pairing of electrons in a solid. In a nucleus, however, unlike in a solid, only a small number of active particles can correlate under the influence of a pairing force. As a consequence, all features of the transition to nuclear superconductivity are softened and it is extremely difficult to see manifestations of the superconducting phase transition. In this work, we develop an exact mapping between the nuclear pairing problem and a classical electrostatic problem in two dimensions. This mapping permits us to recast the problem of nuclear pairing in a completely new setting, one in which the phase transition to superconductivity can be vividly seen. As we will see, the rich phenomenology that appears in the classical problem under appropriate conditions can provide important new insight into the phenomenon of superconductivity in nuclei.

The key to this new view of nuclear pairing comes from the fact that the traditional pairing problem is exactly solvable. Its solution was first presented by Richardson [1–4] in the 1960s in a series of papers. Though recent work has shown how to generalize Richardson’s solution to other systems governed by pairing [5], and there have been recent applications of these generalized models to boson systems [6,7], we will restrict ourselves here to the original fermion problem, for which the hamiltonian takes the familiar form

$$H = \sum_j \varepsilon_j \hat{n}_j - \frac{g}{4} \sum_{jj'} A_j^\dagger A_{j'} , \quad (1)$$

where

$$\hat{n}_j = \sum_m a_{jm}^\dagger a_{jm} \quad , \quad A_j^\dagger = \sum_m a_{jm}^\dagger a_{j\bar{m}}^\dagger = (A_j)^\dagger . \quad (2)$$

Note that the pairing hamiltonian (1) includes a single-particle energy term that splits the set of active levels.

As discussed in ref. [5], the eigenstates and eigenvalues of the pairing problem can be obtained by solving a set of simultaneous eigenvalue equations

$$R_j |\Psi\rangle = r_j |\Psi\rangle , \quad (3)$$

where the R_j are a set of mutually commuting operators that depend on a set of parameters $\{\eta_j, g\}$. The pairing hamiltonian (1) corresponds to the linear combination $H = 2 \sum_j \varepsilon_j R_j + cte$, obtained by choosing the η_j to be the single-particle energies. It is worthwhile to note here that other exactly-solvable pairing hamiltonians can be obtained by making use of the freedom in the parameters η [6,7]. The form of the R operators was first found in Ref. [8], where integrability of the pairing hamiltonian (1) was demonstrated. The eigenvectors of the R operators are equivalent to those originally proposed by Richardson to diagonalize the hamiltonian (1), viz:

$$|\Psi\rangle = \prod_{\alpha=1}^M \left(\sum_j \frac{1}{2\varepsilon_j - e_\alpha} A_j^\dagger \right) |\nu\rangle , \quad (4)$$

where M is the number of collective pairs and $\nu = \sum_i \nu_i$ is the number of unpaired particles, so that the total number of particles is $N = 2M + \nu$. [The ground state corresponds to $\nu = 0$ ($\nu = 1$) for even (odd) systems]. The M unknowns e_α , which are referred to as the *pair energies*, are determined by solving the set of equations (3).

Solving (3) leads to two sets of equations. One is a set of equations for the unknown pair energies e_α (called the Richardson equations),

$$1 + 2g \sum_j \frac{k_j}{2\varepsilon_j - e_\alpha} - 2g \sum_{\beta(\neq\alpha)} \frac{1}{e_\alpha - e_\beta} = 0 , \quad (5)$$

where $k_j = \frac{\nu_j}{2} - \frac{\Omega_j}{2}$ and $\Omega_j = j+1/2$. The quantity $-2k_j$ plays the role of an effective pair degeneracy giving the maximum number of pairs, allowed by the Pauli principle, that the single-particle orbit j can accommodate. The second gives the eigenvalues of the R operators,

$$r_i = k_i \left[1 - g \sum_{j(\neq i)} \frac{k_j}{\varepsilon_i - \varepsilon_j} - 2g \sum_\alpha \frac{1}{2\varepsilon_i - e_\alpha} \right] . \quad (6)$$

The pairing eigenvalues are given by $E = \sum_\alpha e_\alpha$, making clear why the e_α are called the pair energies.

We now discuss how to establish an exact electrostatic analogy for the pairing problem, building on ideas first published by Richardson [9]. To do so, we define an energy functional U

$$U = \frac{1}{2g} \left(\sum_{\alpha} e_{\alpha} + 2 \sum_j k_j \varepsilon_j \right) - \sum_{j\alpha} k_j \ln |2\varepsilon_j - e_{\alpha}| - \frac{1}{2} \sum_{\alpha \neq \beta} \ln |e_{\alpha} - e_{\beta}| - \frac{1}{2} \sum_{i \neq j} k_i k_j \ln |2\varepsilon_i - 2\varepsilon_j|. \quad (7)$$

A similar energy functional has been recently derived from Conformal Field Theory [10]. It is straightforward to verify that the Richardson equation (5) can be obtained by taking derivatives of U with respect to the pair energies e_{α} . Likewise the eigenvalues of the R operators (6) in units of g can be obtained from U by taking derivatives with respect to twice the single-particle energies $2\varepsilon_i$.

In searching for a physical meaning to the energy U , we remind the reader that the Coulomb potential in 2D due to the presence of a unit charge at the origin is given by the solution of the Poisson equation

$$\Delta v(\mathbf{r}) = -2\pi\delta(\mathbf{r}). \quad (8)$$

From this, it is easy to confirm that the Coulomb interaction between two point particles is

$$v(\mathbf{r}_1, \mathbf{r}_2) = -q_1 q_2 \ln |\mathbf{r}_1 - \mathbf{r}_2|, \quad (9)$$

where q_i is the charge and r_i the position of particle i .

Returning to the energy functional (7), we now recognize it as describing a 2D classical electrostatic system of L fixed charges (L is the number of orbitals in the valence space) at positions $2\varepsilon_i$ and with charges k_i (these charges are in general negative for small values of the orbital seniority ν_i), and M free charges located at positions e_{α} and with positive unit charge. The real axis is mapped onto the vertical axis and the imaginary axis onto the horizontal axis. Besides the Coulomb interaction between all charges (the second, third and fourth terms in eq. (7)), there is a uniform electric field in the vertical direction (the first term) whose intensity is inversely proportional to twice the pairing strength, $2g$. Since the fixed charges are related to the single-particle orbitals, we will call them *orbitons*. The free charges associated with the pair energies will be called *pairons*.

Table I: Correspondence between properties of the quantum pairing model and those of the classical two-dimensional electrostatic problem.

Pairing Model	Electrostatic
Single-particle energy: ε_i	Orbiton position: $2\varepsilon_i$
Orbital pair degeneracy: Ω_i	Orbiton charge: k_i
Pair energy: e_{α}	Pairon position: e_{α}
Pairing strength: g	Electric field intensity: $\frac{1}{2g}$

Table I summarizes the translation of the quantum pairing model to the classical electrostatic problem.

Solving the Richardson equations for the pair energies e_{α} is thus completely equivalent to finding the stationary solutions for the pairon positions in the classical electrostatic problem.

Since the orbiton positions are given by the real single-particle energies, they are fixed to lie on the vertical axis. The pairon positions are not of necessity constrained to the vertical axis, but rather must be reflection symmetric around the vertical axis. This reflection symmetry property can be readily seen by performing complex conjugation on the Coulomb energy functional (7). As a consequence, a pairon must either lie on the vertical axis (real pair energies) or must be part of a mirror pair (complex pair energies). The various stationary pairon configurations can be readily traced back to the weakly interacting system ($g \rightarrow 0$). In this limit, the pairons are distributed around the orbitons forming artificial *atoms*. As mentioned above, the number of pairons surrounding the i th orbiton cannot exceed $-2k_i$, which for the ground state (where $\nu = 0$) is Ω_i . Thus, for small g , the ground-state configuration corresponds to distributing the pairons around the lowest-position orbitons consistent with this Pauli constraint. We then let the system evolve gradually with increasing g until we reach its physical value. The excited states can be constructed starting from a different initial pairon configuration and/or by breaking pairs (increasing the seniority ν).

As an example, we now show some results for the ground states of the two semi-magic nuclei ^{114}Sn and the ^{116}Sn . Details on the solution of the Richardson equations (5) can be found in ref. [5]. Table II shows the orbiton positions and charges, which are the same for both nuclei under the usual assumption that the single-neutron energies do not change with neutron number.

Table II: Positions and charges of the orbitons appropriate for a description of the Sn isotopes.

Orbiton	Position	Charge
$d_{5/2}$	0.0	-1.5
$g_{7/2}$	0.44	-2.0
$s_{1/2}$	3.80	-0.5
$d_{3/2}$	4.40	-1.0
$h_{11/2}$	5.60	-3.0

The nucleus ^{114}Sn has 14 valence neutrons and thus in our classical electrostatic analogy seven pairons. In the weak coupling limit, the lowest configuration has three pairons close to the $d_{5/2}$ orbiton and the other four close to the $g_{7/2}$ orbiton. Because of the reflection symmetry property, the four pairons close to the $g_{7/2}$ orbiton form two mirror pairs, whereas the three that are close to the $d_{5/2}$ orbiton form one mirror pair and an odd pairon constrained to the vertical axis.

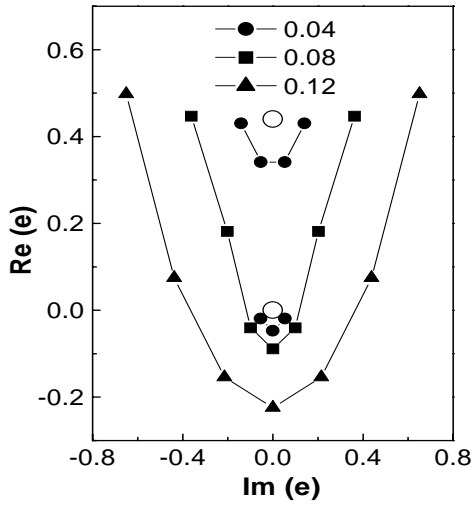


FIG. 1. Two-dimensional representation of the pairon positions in the ^{114}Sn for three selected values of g . The orbitons are represented by open circles; only the lowest two, the $d_{5/2}$ and $g_{7/2}$, are shown at the positions dictated by Table II.

In figure 1 we show the pairon positions in the 2D plane for three selected values of g . [The physical value of g is approximately $-21.7 \text{ MeV}/A = -0.190 \text{ MeV}$.] As seen in the figure, for small values of g the pairons are indeed organized around the two lowest orbitons and form two *atoms*. The other three orbitons correspond to levels that are fairly high in energy and thus lie outside the figure. The fact that the pairons tend to stay in positions below the corresponding orbitons is due to the interplay between the orbiton attraction and the external electric field which points downwards for attractive pairing. As g increases, the atoms expand to reduce the Coulomb interaction energy thereby balancing the reduction in the electric field and there appears a transition from the two isolated atoms to a *cluster* at around $g \sim 0.08$. [Note: In the figure, each pairon is connected by a line to the one that is nearest to it.] We claim that this geometrical transition from atoms to clusters in the classical problem is a direct reflection of the superconducting transition in the quantum problem. The transition from normal to superconducting, which is so difficult to see in the exact quantum results because of the finiteness of the Hilbert space, shows up very clearly and in a highly pictorial fashion through the electrostatic analogy.

In Figure 2, we show the corresponding results for the nucleus ^{116}Sn . At fairly weak coupling ($g = 0.06$ in Fig. 2A), seven of the pairons are organized in two atoms, as in ^{114}Sn , while the eighth lies close to the next orbiton, the $s_{1/2}$. This last pairon is then constrained to move downwards along the vertical axis as g increases.

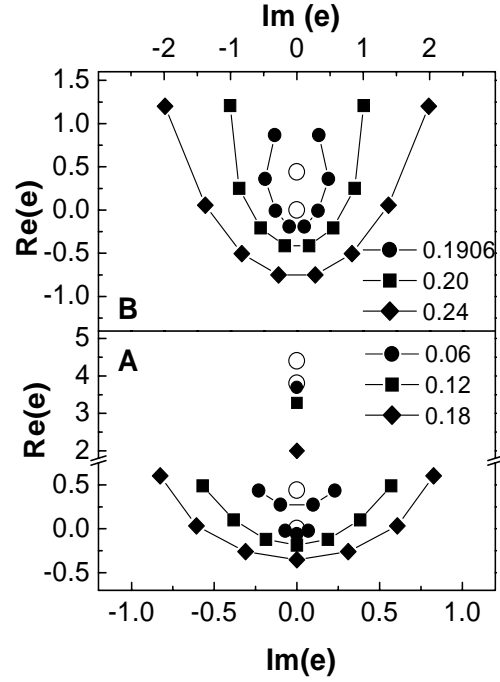


FIG. 2. Two-dimensional representation of the pairon positions in ^{116}Sn for three selected values of g before the collapse (Panel A) and three values of g after the collapse (Panel B). The orbitons are represented by open circles at the positions given in Table II. In Panel A, the energy scale extends high enough to include four orbitons, the $d_{5/2}$, $g_{7/2}$, $s_{1/2}$ and $d_{3/2}$. In Panel B, only the $d_{5/2}$ and $g_{7/2}$ are within scale.

The lowest two atoms display a transition to a cluster, already clearly evident by $g = 0.12$. At this point, however, the last pairon is not yet correlated with the cluster, but rather still approaching it from above. Since it is constrained to be on the vertical axis, there is a critical value of g at which it crosses the $g_{7/2}$ orbiton, which for this problem occurs for $g_c = 0.19044$, very near the physical value of 0.187. At this point, a dramatic phenomenon takes place, as illustrated in Figure 3. To cancel the logarithmic divergence in the energy due to the on-site interaction between the pairon and the $g_{7/2}$ orbiton, all other pairons collapse onto the two lowest orbitons. After the collapse, a new expansion takes place with increasing g (Fig. 2B). Now, however, the two odd pairons associated with the $d_{5/2}$ and the $s_{1/2}$ orbitons form a mirror pair. All eight pairons, now in four mirror pairs, then expand in the 2D plane forming a cluster around the two lowest orbitons. At this point and beyond, all pairs are collectively involved in the quantum superconductivity.

The electrostatic analogy also enables us to understand the interesting pattern of occupation probabilities for ^{116}Sn illustrated in the lower panel of Fig. 4. They were calculated for the exact solution using the Hellmann-Feynman theorem and taking derivatives of the pairing

hamiltonian (1) with respect to the single-particle energies ε_i (details

the exact solution could replace the BCS part of Skyrme HF+BCS codes, and in this way improve their accuracy.

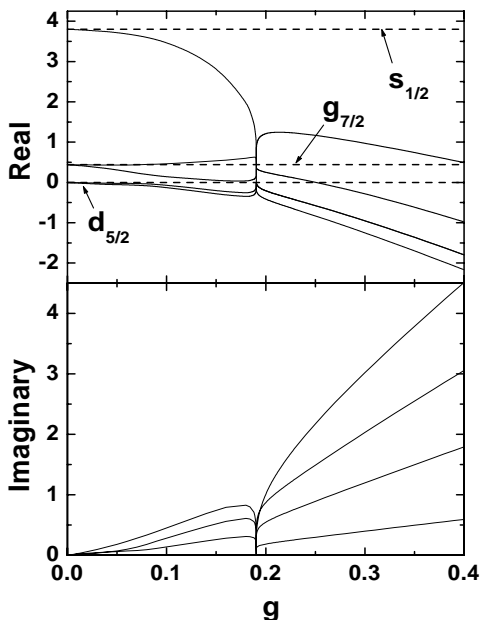


FIG. 3. Real (Upper) and imaginary (Lower) parts of the pair energies for ^{116}Sn as a function of g . Results are shown only for those with $\text{Im}(e_\alpha) \geq 0$.

can be found in [3,4]). Note that as g increases, the occupation of the $s_{1/2}$ orbit behaves in a very dramatic fashion, first decreasing precipitously, then stabilizing and rising together with those of the uppermost $d_{3/2}$ and $h_{11/2}$ orbits. The precipitous drop is associated with the rapid movement of the eighth pairon along the vertical axis, from the $s_{1/2}$ orbiton, with which it was initially connected, towards the $g_{7/2}$ orbiton. As this is taking place, the other seven orbitons behave exactly as for ^{114}Sn , as do the corresponding occupation probabilities (see the upper panel of Fig. 4). Once the collapse takes place, the eighth pairon suddenly becomes part of the cluster. Correspondingly, from this point on all 16 nucleons participate in the superconductivity. For larger values of g , the $s_{1/2}$ orbit behaves much like the two higher ones, gradually increasing its occupation as the collectivity becomes further enhanced. In the large- g limit, all occupation probabilities converge to a value of $1/2$.

For moderate dimensions, like the examples discussed in this letter, the pairing hamiltonian can be exactly diagonalized in a quasispin basis. The Richardson solution discussed here has two particular features not present in other approaches. On the one hand, since the number of Richardson equations (and equivalently the number of unknowns) is the number of fermion pairs, the method can be readily implemented for extremely large model spaces involving many major shells. Thus, for example,

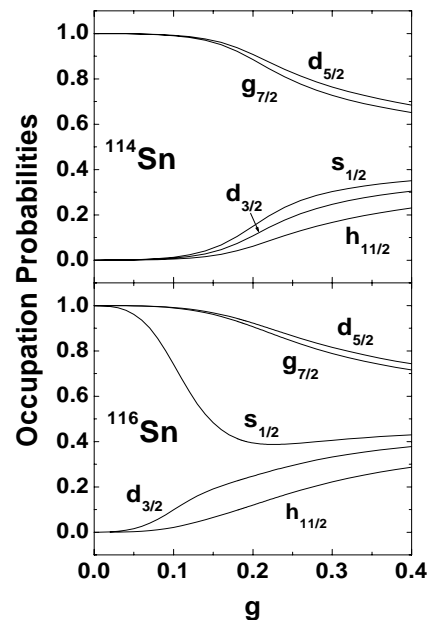


FIG. 4. Calculated occupation probabilities for ^{114}Sn (Upper) and ^{116}Sn (Lower) as a function of g .

Equally important is that the Richardson solution leads naturally to an electrostatic analogy, and this provides useful pictorial insight into pairing phenomena. As we have seen in this work, it makes clear that the superconducting state is realized when collective (Cooper) pairs are developed which involve the cooperative participation of all active orbits with all connection to individual orbits lost. This phenomenon is exhibited in the electrostatic picture as the formation of a pairon cluster. The extensions proposed above in the context of Skyrme HF+BCS calculations would permit the use of the electrostatic analogy to obtain insight into the pairing properties of more complex nuclear systems.

In summary, we have developed in this work an exact mapping of the nuclear pairing model onto the classical two-dimensional electrostatic Coulomb problem. The classical systems that arise by analogy with the Tin region were studied, revealing a rich phenomenology with important implications for our understanding of nuclear superconductivity. Finally, though we focused here on applications to nuclei, the general ideas and methods we presented can be applied to any fermion system governed by pairing correlations.

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